Differentiation and Absolute Continuity

By Gabhan Berry

Supervised by Dr. B. Nasatyr

Department of Mathematics

University of Aberdeen, Scotland

Introduction

In this paper, we study what is meant by *Absolute Continuity*. By the end of the paper, we will have provided the reader with enough understanding to be able to answer questions such as:

> *What is Absolute Continuity?*
> *How is Absolute Continuity different from Continuity?*
> *What important theorems do we have about Absolute Continuity?*

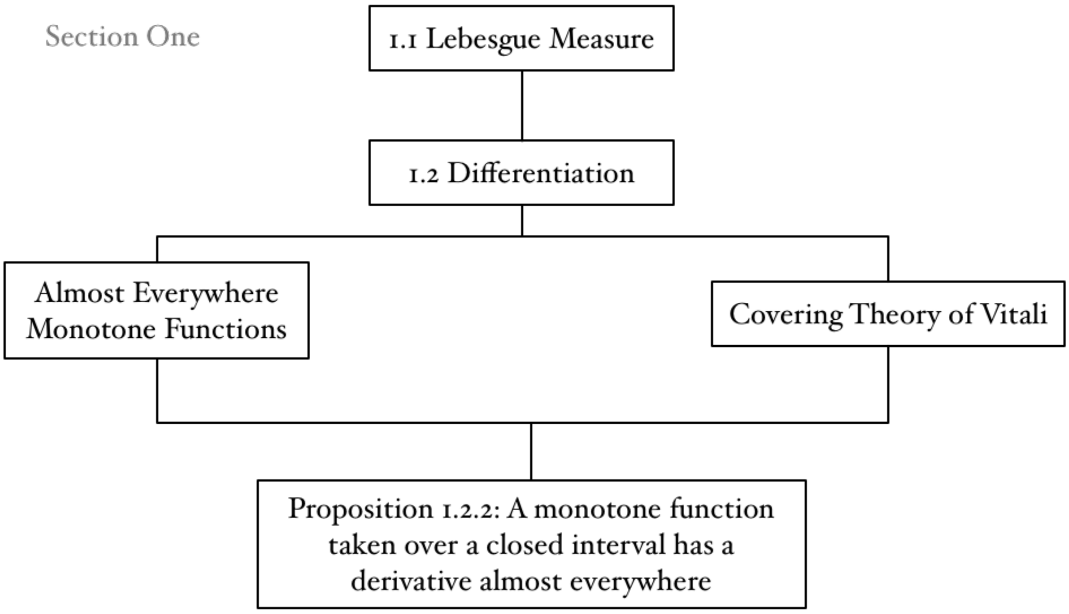
In order to do this (while giving thorough proofs), we need to go deep into the theory of differentiation; and into some other related ideas. We begin, in Section One, with an introduction of the *Lebesgue measure*. This is intended to be a preliminary topic only before we move onto differentiation, where we define the concepts of *monotone functions* and *almost everywhere* before proving that the derivative of a real-valued monotone function exists almost everywhere. This is a large and important (but complex) proof and should be read with some care.

Section Two introduces *absolute continuity*.

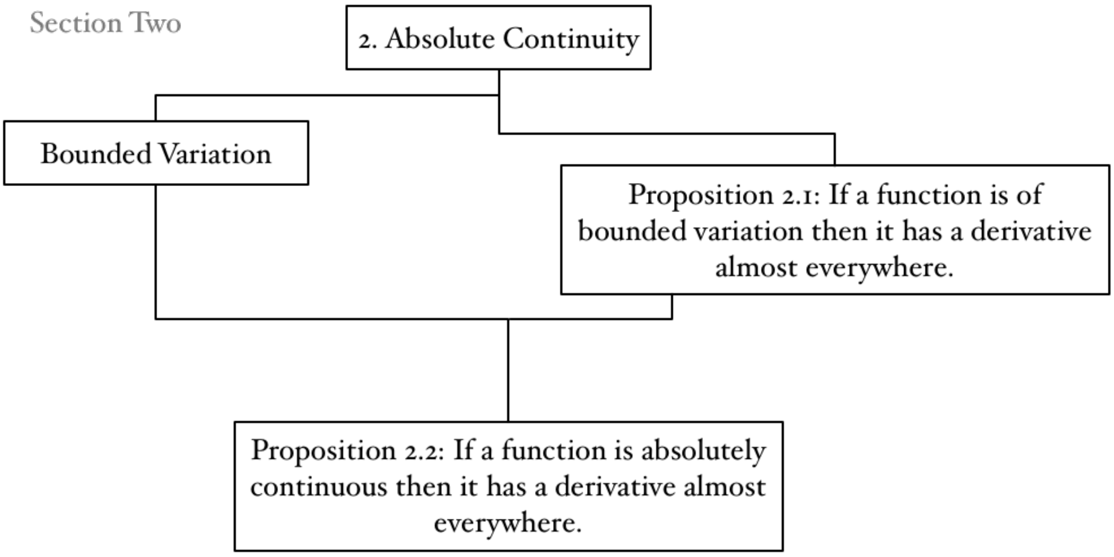
We begin with the definition and then define the concept of *bounded variation*, after which we give two proofs. In the first, we prove that a function of bounded variation has a derivative almost everywhere and, in the second, we prove that an absolutely continuous function has a derivative almost everywhere. We then give an example of a function which is continuous but is not absolutely continuous. Section Two is concluded with a summary of the main points.

This paper contains long and complex proofs. Unfortunately, this has been unavoidable, due to the nature of the topic being covered. These long proofs and the constraints placed upon the length of the paper have forced us to omit an in-depth discussion of the origins of differentiation and continuity.

Section One



Section Two



1.1 Lebesgue Measure

As a preliminary, we will briefly discuss the *Lebesgue Measure*. We will give its definition and a few of its properties, without going into detail. References are provided, for further reading.

Consider the interval $I = [a, b] \in \mathfrak{R}$. We define the *length* of I as $\lambda(I) = b - a$.

We can extend this notion of length to deal with more complicated subsets. Define Γ to be the family of all countable collections of open intervals of $\mathfrak{R}$. For any such collection $\gamma \in \Gamma$, we define the sum:

$$\lambda^*(\gamma) = \sum_{I \in \gamma} \lambda(I)$$

and this is either ∞ or a non-negative real number. Note that the value $\lambda^*(\gamma)$ depends only on γ and not on the order in which the summation is performed.

Now, let E to be an arbitrary subset of $\mathfrak{R}$ and define $\vartheta(E)$ as a subfamily of Γ where:

$$\vartheta(E) = \{\gamma \in \Gamma : \gamma \text{ covers } E\}$$

With this, we can now state a few important concepts that we will use throughout this paper.

__Definition__: The *Lebesgue Outer Measure*, $\mu^*(E)$, of a subset $E \in \mathbf{R}$ is defined as:

$$\mu^*(E) = inf\{\lambda^*(\gamma) : \gamma \in \vartheta(E)\}$$

__Definition:__ A subset $E \in \mathfrak{R}$ is *Lebesgue measurable* if and only if for every subset $X \in \mathfrak{R}$:

$$\mu^*(E) \geq \mu^*(X \cap E) + \mu^*(X - E)$$

Proposition 1.1: For any interval $I \in \mathfrak{R}$, $\mu^*(I) = \lambda(I)$. That is, the Lebesgue outer measure of the interval equals the length of the interval.

We do not provide a proof of Proposition 1.1 but, instead, refer the reader to Real Analysis, Klambauer, page 8.

We will now end this preliminary section by listing some properties of the Lebesgue outer measure.

1. If a subset $E \in \mathfrak{R}$ is Lebesgue measurable, so is its complement $E^c = \mathfrak{R} - E$.
2. If $\mu^*(E) = 0$ then E is Lebesgue measurable.
3. If E_1 and E_2 are Lebesgue measurable sets, so is $E_1 \cup E_2$.

For proofs of these properties, we refer the reader to: for property 1, Real Analysis, Klambauer, page 11, proposition 10; and for properties 2 and 3, Real Analysis, Royden, page 47, lemmas 6 and 7.

1.2 Differentiation

In this section, we will show that the derivative of a monotone function exists *almost everywhere*. But first, we need to establish a few more definitions, beginning with what we mean by *almost everywhere*.

Definition: Let $P(X)$ be a statement about a point x where $x \in E$, a Lebesgue measurable set in $\mathfrak{R}$. We say that $P(x)$ is true *almost everywhere* if and only if the set of points where $P(x)$ is not true has Lebesgue outer measure zero.

Definition: Let A be a subset of $\mathfrak{R}$. An extended real-valued function α defined on A is said to be *non-decreasing* on A if:

$$\alpha(x) \leq \alpha(y) \text{ whenever } x < y \text{ in } A$$

And α is said to be *non-decreasing* on A if:

$$\alpha(y) \geq \alpha(x) \text{ whenever } y > x \text{ in } A$$

And α is said to be *monotone* if it is either non-increasing or non-decreasing.

Bearing this in mind, we give one more definition.

Definition: Let $E \subset \mathfrak{R}$. A collection, Ω, of closed intervals in $\mathfrak{R}$, each having a positive length, is said to be a *Vitali Cover* of E if, for each $x \in E$ and each $\epsilon > 0$, $\exists$ an interval $I \in \Omega$ such that $x \in I$ and $\mu^*(I) < \epsilon$.

In words, what this means is the collection of intervals, Ω, completely covers E. Also, the given condition: that each interval, I, has an arbitrarily small length, means that each point in E can be contained in an arbitrarily small interval.

Now that we have defined the Vitali Cover, we can now establish the *Covering Theorem of Vitali*. Again, due to the constraints placed upon the length of this paper, we offer no proof of this theorem but refer the reader to Real Analysis, Klambauer, page 102.

Proposition 1.2.1: (*The Covering Theorem of Vitali*) Let E be any subset of $\mathfrak{R}$ and let Ω be any non-empty Vitali cover of E. Then there is a mutually disjoint countable family $\{I_n\} \subset \Omega$ such that:

$$\mu^*(E \cap (\cup_{n=1}^{\infty} I_n)^c) = 0$$

where the superscript c denote set complementation relative to $\mathfrak{R}$. Also, if $\mu^*(E) < \infty$ then for each $\epsilon > 0$ there is a mutually disjoint finite family $\{I_1, I_2, \ldots, I_p\}$ such that:

$$\mu^*(E \cap (\cup_{n=1}^{\infty} I_n)^c) < \epsilon$$

Thus far, we have laid out quite a few definitions but it is not clear where we are heading. We needed these definitions and the Covering Theory of Vitali in order to prove the next proposition, which is the main point of this section.

Proposition 1.2.2: Let $[a, b]$ be a closed interval in $\mathfrak{R}$ and let *f* be a real valued monotone function on $[a, b]$. Then, *f* has a finite derivative almost everywhere in the interval $[a, b]$.

Before beginning the proof of Proposition 1.2.2, we need to ensure that the following concepts are firmly understood, because without an understanding of these concepts the proof cannot be understood:

- *Left and right derivatives*
- *Lower and upper left derivatives*[1]
- *Lower and upper right derivatives*

Let $a, \delta \in \mathfrak{R}$ where $\delta > 0$. If *f* is a real-valued function defined on the half-open interval $[a, a + \delta)$, then we define:

The *lower right derivative* as: $D_+f(a) = \underline{lim}_{h\downarrow 0}\dfrac{f(a+h) - f(a)}{h}$

The *upper right derivative* as: $D_+f(a) = \overline{lim}_{h\downarrow 0}\dfrac{f(a+h) - f(a)}{h}$

The *lower left derivative* as: $D_+f(a) = \underline{lim}_{h\uparrow 0}\dfrac{f(a+h) - f(a)}{h}$

[1] The left and right lower and upper derivatives are also called the Dini derivatives, named for Uilisse Dini who introduced them.

The *upper left derivative* as: $D_+f(a) = \overline{lim}_{h\uparrow 0}\dfrac{f(a+h)-f(a)}{h}$

And, the following inequalities always hold:

$$D_+f(a) \leq D^+f(a)$$
$$D\_f(a) \leq D^-f(a)$$

If $D^+f(a) = D_+f(a)$ then f is said to have a *right derivative* at a and we write $f'_+(a)$ for the value of $D^+f(a) = D_+f(a)$. If $D^-f(a) = D\_f(a)$ then f is said to have a *left derivative* at a and we write $f'\_(a)$ for the value of $D^-f(a) = D\_f(a)$.

If both $f'_+(a)$ and $f'\_(a)$ exist, we say that f is *differentiable at a* and we write $f'(a)$ to denote the derivative of f at a. However, we must note that ∞ and $-\infty$ are possible values of $f'(a)$.

With this in hand, we are now ready to prove *Proposition 1.2.2*.

Proof of Proposition 1.2.2: The strategy we will use is as follows. Since we know that the inequality $D_+f(a) \leq D^+f(a)$ always holds, we will define a set which contains the points where $D_+f(a) < D^+f(a)$ and then show that this set has a Lebesgue outer measure of zero, showing, therefore, that $D_+f(a) = D^+f(a)$. Using the same technique, we will show that $D\_f(a) = D^-f(a)$ which will allow us to conclude that $f'(x)$ exists. We will assume that f is non-decreasing. We can do so without losing any generality because, if f was non-increasing, we would just consider $-f$ instead of f.

Let $E = \{x : a \leq x \leq b, D_+f(x) < D^+f(x)\}$, that is, E is the set of points where $D_+f(a) < D^+f(a)$. We want to show that $\mu^*(E) = 0$.

For every pair of positive real numbers w and v where $w < v$, we define the set $E_{w,v} = \{x \in E : D_+f(x) < w < v < D^+f(x)\}$. It is enough to show that $\mu^*(E_{w,v}) = 0$ for all w, v. We will assume the opposite, that is, $\mu^*(E_{w,v}) = \alpha$ for some $\alpha > 0$ and show that doing so leads to a contradiction.

Let ϵ be such that $0 < \epsilon < \dfrac{\alpha(v-w)}{w-2v}$. We choose an open set $G \subset E_{w,v}$ such that $\mu^*(G) < \alpha + \epsilon$. For each $x \in E_{w,v}$ there exists an arbitrarily small positive number h such that $[x, x+h] \subset G \cap [a,b]$ and:

$$f(x+h) - f(x) < wh \tag{1}$$

The family Ω of all such closed intervals is (by definition) a Vitali cover of the set $E_{w,v}$ and, therefore, by Proposition 1.2.1, there exists a finite, pairwise, disjoint subfamily $\left\{[x_i, x_i + h_i]\right\}_{i=1}^{m}$ of Ω such that:

$$\mu^*\left(E_{w,v} \cap \left(\cup_{i=1}^{m} [x_i, x_i + h_i]\right)^c\right) < \epsilon$$

Let , then we have:

$$\mu^*(E_{w,v} \cap V^c) < \epsilon \tag{2}$$

Now we note that $V \subset G$ (recall that V is the union of the open intervals of the form $(x_i, x_i + h_i)$, whereas G is defined to contain the union of closed intervals of the form $[x_i, x_i + h_i]$, hence, $V \subset G$). This gives us:

$$\sum_{i=1}^{m} \left(f(x_i + h_i) - f(x_i)\right) < w \sum_{i=1}^{m} h_i < w(\alpha + \epsilon) \tag{3}$$

Equation (3) is formed by using (1) not just for one interval but for all i intervals; in conjunction with the fact that the length of V is the sum of the union of intervals $[x_i, x_i + h_i]$ for $0 \leq i \leq m$. Which is to say that $\mu^*(V) = \sum_{i=1}^{m} h_i$. So, we have $f(x_i + h_i) - f(x_i) < wh_i$ for each value of i, thus: $\sum_{i=1}^{m} f(x_i + h_i) - f(x_i) < w\mu^*(V)$. Since $V \subset G$ we know that $\mu^*(V) \leq \mu^*(G)$ and so:

$$\implies \sum_{i=1}^{m} f(x_i + h_i) - f(x_i) < w\mu^*(V) \leq w\mu^*(G)$$

$$\implies \sum_{i=1}^{m} f(x_i + h_i) - f(x_i) < w \sum_{i=1}^{m} h_i < w(\alpha + \epsilon),$$ which is equation (3).

Continuing, we note for each $y \in E_{w,v} \cap V$ there exists an arbitrarily small, positive number, k, such that $[y, y + k] \subset V$ and such that:

$$f(y + k) - f(y) > vk \tag{4}$$

The collection of all such closed intervals is a Vitali Cover of the set $E_{w,v} \cap V$ and so, by Proposition 1.2.1, there is a finite, pairwise disjoint family of intervals $\left\{ [y_j, y_j + k_j] \right\}_{j=1}^{n}$ such that:

$$\mu^* \left(E_{w,v} \cap V \cap \left(\cup_{j=1}^{n} [y_j, y_j + k_j] \right)^c \right) < \epsilon \tag{4.1}$$

This, along with equation (2) implies that:

$$\alpha = \mu^*(E_{w,v}) \leq \mu^*(E_{w,v} \cap V^c) + \mu^*(E_{w,v} \cap V) < \epsilon + (\epsilon + \sum_{j=1}^{n} k_j) \tag{5}$$

To see how we arrived at (5), note that (2) tells us that $\mu^*(E_{w,v} \cap V^c) < \epsilon$, so we need to show only how we can say that $\mu^*(E_{w,v} \cap V) < \sum_{j=1}^{n} k_j + \epsilon$. We can rewrite (4.1), by letting $F = E_{w,v} \cap V$ and $G = \cup_{j=1}^{n} [y_j, y_j + k_j]$. Therefore, (4.1) becomes $\mu^*(F \cap G^c) < \epsilon$ and, to arrive at (5), we need to show that $\mu^*(F) < \epsilon + \sum_{j=1}^{n} k_j$.

By definition $\mu^*(F) = \mu * (F \cap G) + \mu^*(F \cap G^C)$. Since $\mu^*(F \cap G) < \mu^*(G)$ and $\mu^*(F \cap G^c) < \epsilon$, we have:

$$\mu^*(F) = \mu * (F \cap G) + \mu^*(F \cap G^C) < \mu^*(G) + \epsilon$$

And we know that $\mu^*(G) = \sum_{j=1}^{n} k_j$ so, putting all this together, we get:

$$\mu^*(E_{w,v}) \leq \mu^*(E_{w,v} \cap V^c) + \mu^*(E_{w,v} \cap V) < \epsilon + (\epsilon + \sum_{j=1}^{n} k_j)$$

which is equation (5).

Next, we note that for each value of j we have equation (4) and if we sum over the n intervals we get: $v \sum_{j=1}^{n} k_j < \sum_{j=1}^{n} \Big(f(y_j + k_j) - f(y_j)\Big)$. From (5) we see $\alpha < 2\epsilon + \sum_{j=1}^{n} k_j$ which we can rewrite as $\alpha - 2\epsilon < \sum_{j=1}^{n} k_j$. Multiplying both sides by v gives us $v(\alpha - 2\epsilon) < v \sum_{j=1}^{n} k_j$ which tells us:

$$v(\alpha - 2\epsilon) < v \sum_{j=1}^{n} k_j < \sum_{j=1}^{n} \Big(f(y_j + k_j) - f(y_j)\Big) \tag{6}$$

The previously defined collection of intervals $[x_i, x_i + h_i]$ is a Vitali Cover of the set $E_{w,v}$, so, each of the intervals in contained within $E_{w,v}$. That is, $[x_i, x_i + h_i] \subset E_{w,v} \, \forall i$. Also, the collection of intervals $[y_j, y_j + k_j]$ is a Vitali Cover of the set $E_{w,v} \cap V$ and, therefore, each of these intervals is contained within $E_{w,v} \cap V$, giving us $[y_j, y_j + k_j] \subset E_{w,v} \cap V \; \forall i$. However, the intervals $[y_j, y_j + k_j]$ do not cover $E_{w,v}$, whereas, the intervals $[x_i, x_i + h_i]$ do. Thus, it follows that, the union of the intervals $[y_j, y_j + k_j]$ is contained within the union of the intervals $[x_i, x_i + h_i]$, which we write as:

$$\cup_{j=1}^{n} [y_j, y_j + k_j] \subset \cup_{i=1}^{m} [x_i, x_i + h_i]$$

Given this and the fact that f is a non-decreasing function, we have:

$$\sum_{j=1}^{n} \Big(f(y_j + k_j) - f(y_j) \Big) \leq \sum_{i=1}^{m} \big(f(x_i + h_i) - f(x_i) \big) \tag{7}$$

Now, we combine equation (7) with equations (3) and (6):

$$\sum_{i=1}^{m} \big(f(x_i + h_i) - f(x_i) \big) < w \sum_{i=1}^{m} h_i < w(\alpha + \epsilon) \tag{3}$$

$$v(\alpha - 2\epsilon) < v \sum_{j=1}^{n} k_j < \sum_{j=1}^{n} \Big(f(y_j + k_j) - f(y_j) \Big) \tag{6}$$

to get:

$$v(\alpha - 2\epsilon) < v \sum_{j=1}^{n} k_j < \sum_{j=1}^{n} \Big(f(y_j + k_j) - f(y_j) \Big) \leq \sum_{i=1}^{m} \big(f(x_i + h_i) - f(x_i) \big)$$

$$\implies v(\alpha - 2\epsilon) < w \sum_{i=1}^{n} h_i < w(\alpha + \epsilon)$$

We can rearrange this inequality to $\epsilon < \dfrac{\alpha(w-v)}{-2v-w}$. But our definition of ϵ states that $\epsilon < \dfrac{\alpha(v-w)}{w-2v}$. Thus, we have arrived at a contradiction. Since our initial assumption was that $\mu^*(E) \neq 0$ and this assumption led to a contradiction about the value of ϵ, we can now say that we have proved that $\mu^*(E) = 0$ and, thus, that $f'_+(x)$ exists almost everywhere.

Using the same method, but with a slightly different definition of E, we get that $f'_-(x)$ exists almost everywhere. This part of the proof is excluded because it is the same as what we just did. All that remains to do to complete the proof is to show that the set B of all points $x \in (a, b)$ for which $f'(x) = \infty$ has Lebesgue outer measure zero.

Let β be an arbitrary positive number. For each $x \in B$ there exists an arbitrarily small number h such that $[x, x+h] \subset (a, b)$ and:

$$f(x+h) - f(x) > \beta h \tag{7}$$

By Proposition 1.2.1 there exists a countable pairwise disjoint family $\{[x_n, x_n + h_n]\}$ such that:

$$\mu^* \left(B \cap \left(\cup_n [x_n, x_n + h_n]\right)^c\right) = 0$$

Now we apply (7) to all of the intervals in $\{[x_n, x_n + h_n]\}$ to get $\beta \sum_n h_n < \sum_n \left(f(x_n, x_n + h_n) - f(x_n)\right)$. Note that $\sum_n h_n \leq (b-a)$ because $(b-a)$ is the length of the interval and the h's are the subdivisions of the interval. Therefore, we get:

$$\beta \sum_n h_n < \sum_n \left(f(x_n, x_n + h_n) - f(x_n)\right) \leq f(b) - f(a) \tag{8}$$

The inequality $\mu^*\left(B \cap \left(\cup_n [x_n, x_n + h_n]\right)^c\right) = 0$ tells us that there are no elements of B in the set $\left(\cup_n [x_n, x_n + h_n]\right)^c$ and so all of the elements of B must be contained in $\cup_n [x_n, x_n + h_n]$ and, therefore, $\mu^*(B) \leq \mu^*(\cup_n [x_n, x_n + h_n])$. Putting this into (8) we get:

$$\beta\mu^*(B) \leq \beta\mu^*(\cup_n [x_n, x_n + h_n]) \leq \beta \sum_n h_n < \sum_n \left(f(x_n, x_n + h_n) - f(x_n)\right) \leq f(b) - f(a)$$

$$\implies \beta\mu^*(B) \leq f(b) - f(a) \;\; \forall \beta \in \mathfrak{R}$$

Since β was picked arbitrarily, this implies that $\mu^*(B) = 0$. This concludes the proof of Proposition 1.2.2.

□

We end this section with a simple example to illustrate Proposition 1.2.2. Consider the function $f(x) = x$ taken over the interval $[a, b] \in \mathfrak{R}$. For any two points $d, c \in [a, b]$ where $d < c$, we know that $f(d) < f(c)$. Therefore, f is non-decreasing (actually, it is strictly increasing) and so f is monotone. The interval $[a, b] \in \mathfrak{R}$ is both closed and real valued. Therefore, Proposition 1.2.2 applies and tells us that f has a finite derivative almost everywhere. Let's check. $f'(x) = 1 \;\; \forall x \in [a, b]$ and the set where this is not true is empty.

2. Absolute Continuity

In this section we talk about functions that are *absolutely continuous*. In the previous section, we showed that a monotone, real valued function taken over a closed interval is differentiable almost everywhere in the interval. Now we will show that is a function is absolutely continuous over a closed interval, then it too is differentiable almost everywhere in the interval. We will also highlight the difference between a continuous function and an absolutely continuous function, to show that absolute continuity is a much stronger condition.

Definition: A real valued function f defined on $[a, b]$ is said to be ***absolutely continuous*** on the interval $[a, b]$ if, given $\epsilon > 0$, there is a $\delta > 0$ such that:

$$\sum_{i=1}^{n} \left| f(x'_i) - f(x_i) \right| < \epsilon$$

for every finite collection $\{(x_i, x'_i)\}$ of non-overlapping intervals where:

$$\sum_{i=1}^{n} \left| x'_i - x_i \right| < \delta$$

Now, we introduce the notion of *Bounded Variation*.

Definition: Let f be a real valued function defined on $[a, b]$. We set:

$$V_a^b f = sup \left\{ \sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right| : a = x_0 < x_1 < \ldots < x_n = b \right\}$$

where the number $V_a^b f$ is called the *Total Variation* of f over $[a, b]$. If $V_a^b f < \infty$ then f is said to have *Bounded Variation* over $[a, b]$.

Now consider the following proposition, which we shall prove.

Proposition 2.1: If f is a real valued function of bounded variation, then f has a derivative almost everywhere.

Proof of Proposition 2.1: We note that a function of bounded variation, $f(x)$, is the difference between two non-decreasing functions:

$$f(x) = V_a^x f - (V_a^x f - f(x))$$

If these two functions, $V_a^x f$ and $(V_a^x f - f(x))$ are monotone then we can apply Proposition 1.2.2 to get that the derivatives of these functions exist almost everywhere and, therefore, the derivative of f also exits almost everywhere. Thus, we need to prove that $V_a^x f$ and $(V_a^x f - f(x))$ are indeed monotone functions.

Let $g = V_a^x f$ and $h = V_a^x f - f(x)$ and also $a < \beta < b$. We need to show that $g(\beta) \le g(b)$ and that $h(\beta) \le h(b)$. First, let's deal with $g(\beta) \le g(b)$.

$$g(b) - g(\beta) = V_a^b f - V_a^\beta f = V_a^\beta f + V_\beta^b f - V_a^\beta f = V_\beta^b f > 0$$
$$\therefore g(\beta) < g(b)$$

And now, we need to show that $h(\beta) \le h(b)$.

$$\begin{aligned} h(b) - h(\beta) &= V_a^b f - f(b) - (V_a^\beta f - f(\beta)) \\ &= V_a^b f - V_a^\beta f + f(\beta) - f(b) \\ &= V_\beta^b f + f(\beta) - f(b) \end{aligned}$$

The variation of a function over an interval is greater than or equal to the difference of the values of the function at the end points. That is:

$$\begin{aligned} &V_\beta^b f \ge f(\beta) - f(b) \\ \implies &V_\beta^b f + f(\beta) - f(b) \ge 0 \\ \therefore\ &h(\beta) \le h(b) \end{aligned}$$

Thus, we have shown that both g and h are monotone and can conclude that we have indeed proved Proposition 2.1 to be true. □

Now that we have proved Proposition 2.1, we can now prove that an absolutely continuous function has a derivative almost everywhere.

Proposition 2.2: If f is absolutely continuous on the interval $[a, b]$ then it has a finite derivative almost everywhere on $[a, b]$.

Proof: All we have to do here is show that f is of bounded variation and then use Proposition 2.1 to get that f has a derivative almost everywhere.

Since f is absolutely continuous, we can pick a δ such that:

$$\sum_{k=1}^{m} \left|(c_{k+1}, c_k)\right| < \delta \text{ and } \sum_{k=1}^{m} \left|f(c_{k+1}) - f(c_k)\right| < 1$$

We can suppose without loss of generality that $c_{k+1} - c_k < \delta$ for all k. That is, we are splitting up the interval $[a, b]$ into m intervals, each of length less than δ. Over each of these intervals we have $\sum \left|f(c_{k+1}) - f(c_k)\right| < 1$ because they are each of length less than δ. So, we have m intervals with $m \leq \frac{2(b-a)}{\delta}$ (if each interval's length was exactly δ then we would have $\frac{b-a}{\delta}$ intervals, but the lentgh of each interval is less than δ). Therefore, we have $\sum \left|f(c_{k+1}) - f(c_k)\right| \leq V_a^b f \leq m \leq \frac{2(b-a)}{\delta} < \infty$. Thus we have shown that f is of bounded variation. □

Proposition 2.3: Let f be a non-negative function which is integrable over a set E. Then given $\epsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $\mu^*(A) < \delta$ we have $\int_A f < \epsilon$.

The proof of Proposition 2.3 is omitted from this paper. The reader is referred to Real Analysis, Klambauer, p63.

Let the set A be as defined in Proposition 2.3 and also let $A = \cup [c_i, c_{i+1}]$ and define a function F as $F = \int f$, that is, $F' = f$, where f is as defined in Proposition 2.3. For any interval in A we have $\int_{c_i}^{c_{i+1}} f = F(c_{i+1}) - F(c_i)$. Integrating over all such intervals in A gives: $\int_A f = \sum \left(F(c_{i+1}) - F(c_i)\right)$. Noting that because f is a non-negative function and that for any function, f, we have $\left|\int F\right| \le \int |f|$, we can say $\int_A f = \sum \left(F(c_{i+1}) - F(c_i)\right) \le \left|\int_A f\right| \le \int_A |f| < \epsilon$. That is, $\sum \left|\left(F(c_{i+1}) - F(c_i)\right)\right| < \epsilon$ and by definition $\mu^*(A) < \delta$ and so $\sum \left|(c_{i+1}, c_i)\right| < \delta$. Thus, F is absolutely continuous on A.

What this is telling us is that, if we have a function F, which we know is absolutely continuous, then we also have: $F(x) = \int_a^x F'(x)\,dx$. That is, F is equal to the integral of its own derivative. So, we now have:

$$F(x) = \int_a^x F'(x)\,dx \iff \text{absolutely continuity} \implies \text{continuity} \qquad (2.4)$$

However, it is not clear that the converse is not true. In order to show this, we will discuss an example where a function is continuous but is not absolutely continuous. This example will highlight the difference between absolute continuity and continuity and show us that absolute continuity is the much stronger condition. In order to get an example, we have to do some extra work. We will start by defining the *Cantor Set*.

The *Cantor Set, C,* is a subset of $\Re$, in the interval [0,1]. That is $C \subseteq [0,1] \subset \Re$. We define the Cantor Set by starting with the set [0,1] and then reducing its size by removing certain intervals.

First, we remove the interval $\left(\frac{1}{3},\frac{2}{3}\right)$ from [0,1]. We call what's left C_1. Next, we remove the middle thirds from each of the two intervals in C_1. That is, we remove $\left(\frac{1}{9},\frac{2}{9}\right)$ from $\left[0,\frac{1}{3}\right]$ and we remove $\left(\frac{7}{9},\frac{8}{9}\right)$from $\left[\frac{2}{3},1\right]$. We call what's left, C_2. We continue in this fashion and, after doing this n times, we call the set we are left with C_n.

<u>*Definition:*</u> The *Cantor Set* is defined to be the set $C = \bigcap_{n=1}^{\infty} C_n$.

At step n, C_n is composed of 2^n intervals, each of length $\frac{1}{3^n}$. So, $\lambda^*(C_n) = 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$. As $n \to \infty$, $\left(\frac{2}{3}\right)^n \to 0$, so $\mu^*(C) = 0$ i.e., the Cantor set has a Lebesgue Outer Measure of zero.

We can express any $x \in [0,1]$ as the decimal $0.a_1a_2a_3\ldots$

$$\implies x = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000}\ldots$$
$$\implies x = \textstyle\sum_{i=1}^{\infty} \frac{a_i}{10^i}$$

where $a_i \in \{0,1,...,9\}$

However, x can also be expressed as a ternary: $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$ where $a_i \in \{0,1,2\}$. For example, if we have the number 0.222, the first 2 gives the number of thirds, the second 2 gives the number of ninths and the third 2 gives the number of twenty-sevenths.

If $x \in C_1$ then $x \in \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$. Suppose that any $x \in C_1$ can be expressed as a ternary. We then get that $a_1 \neq 1$. By the same logic, if $x \in C_2$ then $a_1 \neq 1$ and $a_2 \neq 1$.

Note that $\frac{1}{3} \in C_1$ can be expressed as 0.1 or as 0.022222... The point being that the ternary representation is not unique, but it can always be expressed such that $a_1 \neq 1$. Further, we can continue in this manner for all C_n and, therefore, $x \in C \iff a_i \neq 1 \ \forall \ i \in [1,n]$. So, we now have a nice way of defining C.

$$C = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{3^i} \ \mid \ a_i \in \{0,2\} \right\}$$

Remember that we are looking for a function which is continuous but not absolutely continuous. So we will now define a function $f : \mathfrak{R} \to \mathfrak{R}$ as:

$$f(x) = \begin{cases} \sum_{i=0}^{\infty} \frac{\frac{a_i}{2}}{2^i} & \text{if } \ x \in C \\ 0 & \text{if } \ x < 0 \\ 1 & \text{if } \ x > 1 \\ G & \text{if } \ x \in [0,1] \backslash C \end{cases}$$

Notice that in base 3, $f(0.02222...) = \frac{1}{2}$ and $f(0.2) = \frac{1}{2}$. Indeed, if we remove any middle third the value of f at the end points is equal. Also, the middle thirds are not in C, so we may as well define f to be constant when taken at these values. That is, if $x \in [0,1] \backslash C$ then $f(x) = G$, where G is some constant. For example, $f = G = \frac{1}{2}$ on the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$ and $f = G = \frac{1}{4}$ on the interval $\left[\frac{1}{9}, \frac{2}{9}\right]$. So, now that we know how to define f we need to look at the continuity of f.

Note that $f(1) = 1$ and $f(0) = 0$ and that $f'(1) = f'(0) = 0$. Therefore, f is continuous at these points.

If $x \notin C$ then $f'(x) = 0$. However, remember that $\mu^*(C) = 0$ and therefore $f'(x) = 0$ must be true almost everywhere, because the set where $f'(x) \neq 0$ (i.e, the set C) has a Lebesgue Outer Measure of zero. Therefore, f is continuous almost everywhere. Now, suppose that f is also absolutely continuous. We would then be able to apply 2.4 to say $f(x) = \int_a^x f'(x)\, dx$. Applying this to our function we get $f(x) = \int_a^x 0\, dx = 0$. However, we know from our previous statements about f that f is not 0 on the interval. Thus, we have a contradiction. So, we can conclude that f is indeed a function that is continuous but that is not absolutely continuous.

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